Hydraulic Inverse Modeling Using Total-Variation Regularization with Relaxed Variable-Splitting

Youzuo Lin, Velimir V. Vesselinov, Daniel O'Malley, and Brendt Wohlberg

Los Alamos National Laboratory, Los Alamos, NM 87545, USA

Introduction

- Inverse modeling seeks model parameters given a set of observed-state variables. Typically, inverse analyses are applied to characterize aquifer heterogeneity where the hydraulic permeability is estimated throughout the model domain.
- For many practical hydrogeological problems, because the data coverage is limited, the inversion can be ill-posed and unstable.
- To stabilize the inversion, regularization techniques can be employed to eliminate the ill-posedness.
- The most commonly used type of regularization include Tikhonov and Total-Variation (TV). The hydraulic tomographic analyses of aquifer heterogeneity with Tikhonov regularization tends to yield smoothed inversion results, while the ones with TV regularization can preserve the sharp contrast between low and high permeability regions.
- However, hydraulic inverse modeling with the conventional TV regularization can be computationally unstable and yield unwanted artifacts because of the non-differentiability of the TV norm.
- We have developed a novel hydraulic inverse modeling method using a TV regularization with relaxed variable-splitting scheme to preserve sharp interfaces in piecewise-constant structures and improve the accuracy of inversion.
- We implement our new inversion method using Julia in the MADS computational framework (**http://madsjulia.lanl.gov/**), which can be downloaded at **git@gitlab.com:mads/Mads.jl.git**.

Inverse Hydrogeological Modeling

- We consider a 2-dimensional steady-state groundwater flow equation on a square domain $[a, b] \times [c, d]$,

$$\begin{aligned} \nabla \cdot (T \nabla h) &= g \\ g(x, y) &= 0 \\ \left.\frac{\partial h}{\partial x}\right|_{a,y} &= \left.\frac{\partial h}{\partial x}\right|_{b,y} = 0 \\ h(x, c) = 0&, h(x, d) = 1 \end{aligned}$$

  where h is the hydraulic head, T is the transmissivity and g is a source/sink (here, set to zero).
- The problem of hydrogeologic inverse modeling is posed as a minimization problem,

$$\hat{\mathbf{m}} = \arg\min_{\mathbf{m}} \left\{ \|\mathbf{d} - f(\mathbf{m})\|_2^2 \right\},$$

  where $\mathbf{d}$ represents a recorded hydraulic head dataset and $\mathbf{m}$ is the inverted model parameter, $\|\mathbf{d} - f(\mathbf{m})\|_2^2$ measures the data misfit, and $\|\cdot\|_2$ stands for the L_2 norm.

Regularization Theory

- Inverse modeling with general regularization term can be posed as,

$$\begin{aligned} \hat{\mathbf{m}} &= \arg\min_{\mathbf{m}} \left\{ l(\mathbf{m}) \right\} \\ &= \arg\min_{\mathbf{m}} \left\{ \|\mathbf{d} - f(\mathbf{m})\|_2^2 + \lambda \mathcal{R}(\mathbf{m}) \right\}, \end{aligned}$$

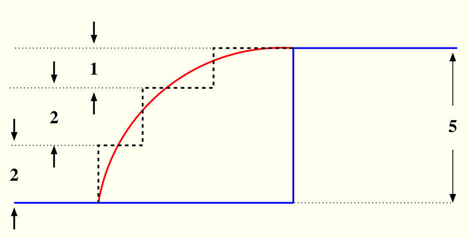
  where $\mathcal{R}(\mathbf{m})$ is a general regularization term and the parameter λ is a parameter controlling the amount of regularization in the inversion.
- Specific Regularization Schemes
  - Total-Variation (TV): $\mathrm{R}(\mathbf{m}) = \|\nabla \mathbf{m}\|_1 = \sum_i |(\delta m)_i|$, (1-D)

    Best suited for reconstructing piecewise-constant functions, computationally expensive
  - Tikhonov (TK): $\mathrm{R}(\mathbf{m}) = \|L\,\mathbf{m}\|_2 = \sum_i (\delta m)_i^2$, (1-D)

    Best suited for reconstructing smooth functions, computationally cheap

  $* \ \mathrm{TV}_{step} = 5$; $\mathrm{TV}_{smooth} = 2 + 2 + 1 = 5$.

  $* \ \mathrm{TK}_{step} = 5^2 = 25$; $\mathrm{TK}_{smooth} = 2^2 + 2^2 + 1 = \mathbf{9} \leftarrow$.



Total Variation Regularization with Relaxed Variable-Splitting Scheme

- The misfit function of hydraulic inverse modeling using total-variation regularization with relaxed variable-splitting is:

$$E(\mathbf{m}, \mathbf{u}) = \min_{\mathbf{m}, \mathbf{u}} \left\{ \|\mathbf{d} - f(\mathbf{m})\|_2^2 + \alpha \|\mathbf{m} - \mathbf{u}\|_2^2 + \beta \|\nabla \mathbf{u}\|_1 \right\},$$

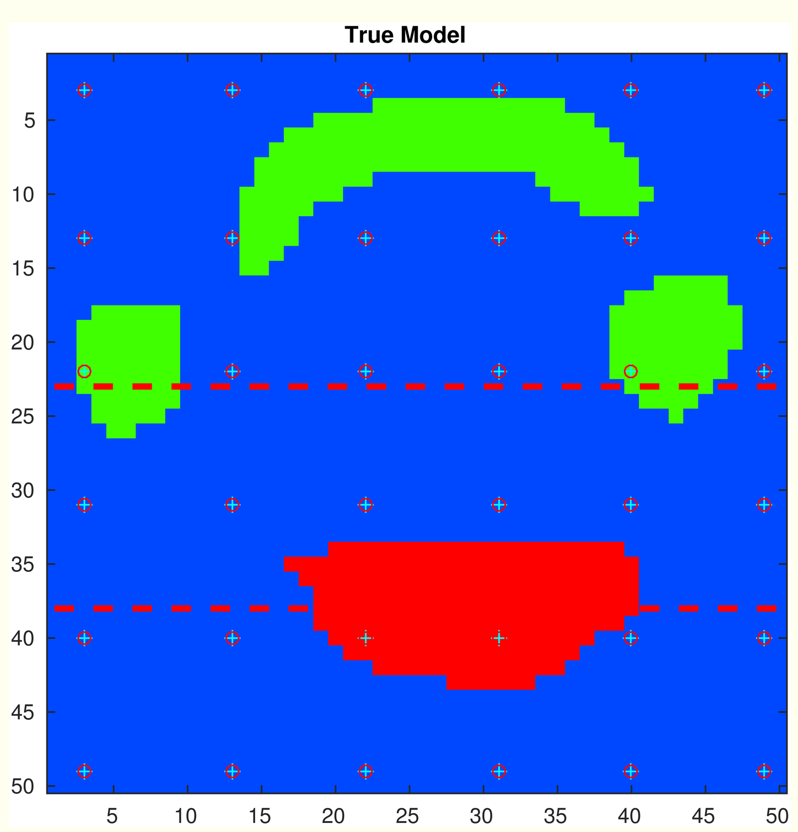
  where $\|\nabla \cdot\|_1$ is the total-variation (TV) term, and $||\cdot||_2$ stands for L_2 norm.
- We employ an alternating-minimization algorithm to solve the above double minimization optimization problem.

$$\begin{cases} \mathbf{m}^{(k)} = \underset{\mathbf{m}}{\operatorname{argmin}} \left\{ \|\mathbf{d} - f(\mathbf{m})\|_2^2 + \alpha \left\| \mathbf{m} - \mathbf{u}^{(k-1)} \right\|_2^2 \right\} \\ \mathbf{u}^{(k)} = \underset{\mathbf{u}}{\operatorname{argmin}} \left\{ \left\| \mathbf{m}^{(k)} - \mathbf{u} \right\|_2^2 + \beta \|\nabla \mathbf{u}\|_1 \right\} \end{cases},$$

  for iteration step $k = 1, 2, \cdots$.

Problem Setup and Model Discretization

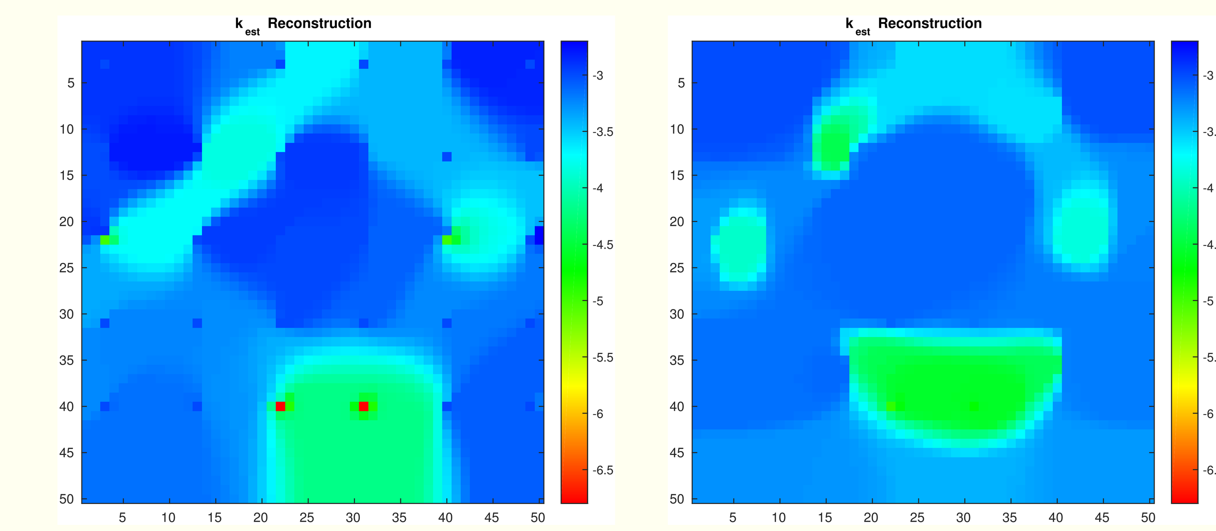
- The reference problem is steady-state groundwater flow on the square domain, $[0, 1] \times [0, 1]$, with fixed hydraulic head at $y = 0$ and $y = 1$, zero flux boundaries at $x = 0$ and $x = 1$, and zero recharge.
- We run the tests on a Linux desktop with 32 cores of 2.0 GHz Intel Xeon E5-2650 CPU, and 16.0 GB memory.
- The groundwater flow equation is solved using the finite difference method on a uniform grid. The parameter grids are composed of horizontal and vertical transmissivity nodes (as are illustrated in figure below).
- The true model is created similar to the one from Lee and Kitanidis' work (*"Bayesian inversion with total variation prior for discrete geologic structure identification"*, WRR, Vol. 49, P. 7658–7669, 2013.)
- The dimension of the true model is 50×50. The crosses ("X") are the hydraulic-head observation points (wells).
- Two low-permeable geologic facies are included in the true model representing: sand (green) and clay (red). The background is highly permeable (gravel; blue). The permeability within all the three facies is assumed to be uniform.
- Two horizontal profiles indicated by the red dotted line will be used to compare the results obtained using different inversion methods.



True Model

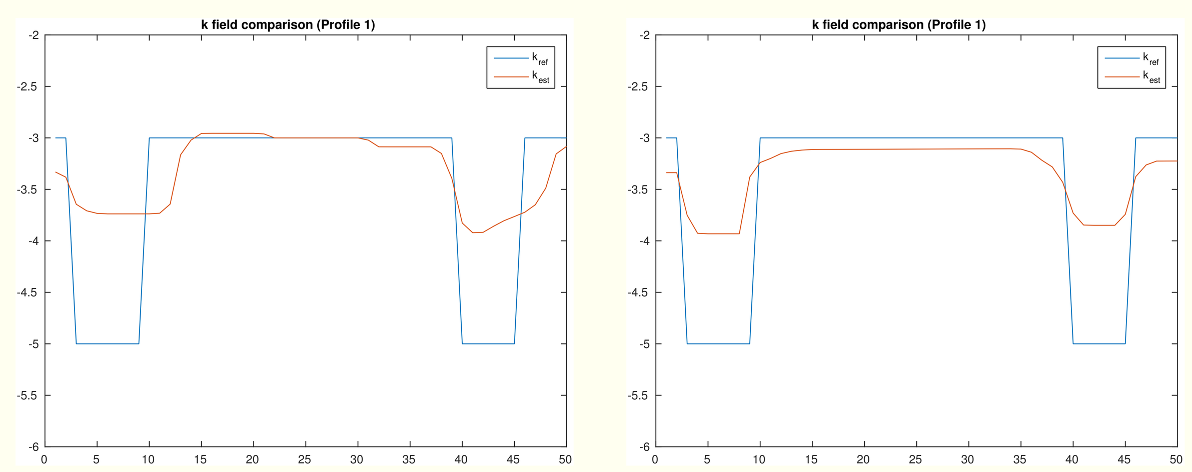
Numerical Results

- **The reference method is hydraulic inverse modeling with conventional TV regularization.**
- **Inversion Results**



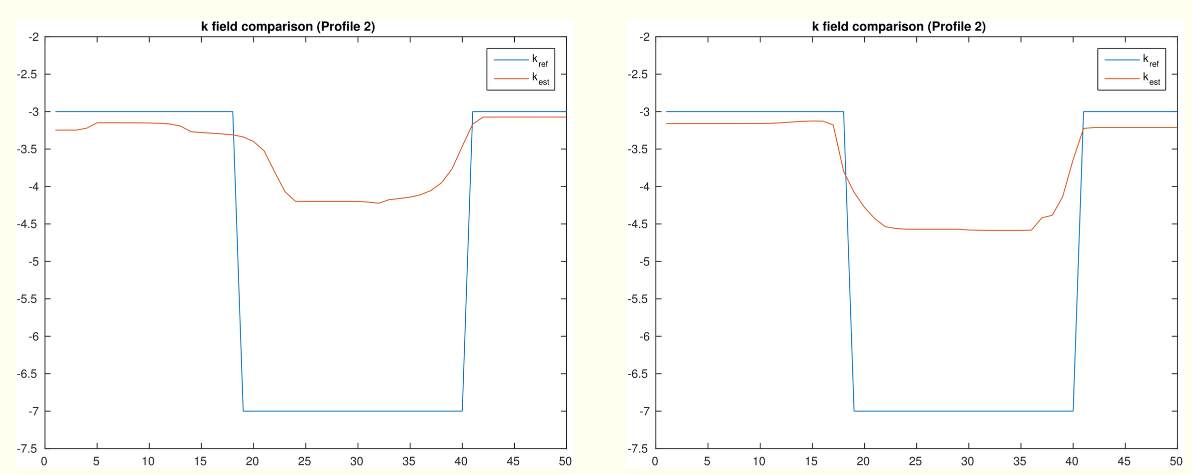
(a) Reference Method (b) Our Method

- **Comparison of Horizontal Profiles**



(a) Reference Method (b) Our Method

Horizontal profiles going through the sand regions.



(a) Reference Method (b) Our Method

Horizontal profiles going through the clay region.

- Our method yields more accurate results with much better preserved facies boundaries than the one obtained using the reference method. This is particularly true in reconstruction of the clay region.
- Our method also yields less spatial artifacts, which can be observed in the profile comparison.

Conclusions

- We have developed a hydraulic inverse modeling using total-variation regularization with relaxed variable-splitting.
- Our numerical examples using synthetic data show that our new methods not only preserve sharp interfaces between facies with contrasting permeabilities, but also significantly improve the accuracy of the inversion. Therefore, our method has great potential in characterizing the subsurface heterogeneity problems.

Acknowledgments: Youzuo Lin, Velimir V. Vesselinov, Daniel O'Malley, and Brendt Wohlberg were support by Los Alamos National Laboratory Environmental Programs Projects. In addition, Velimir V. Vesselinov was supported by the DOE DiaMonD project.